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# Numerical Study of the Characteristics of Shock and Rarefaction Waves for Nonlinear Wave Equation

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**Abstract:** In everyday life human faces shock waves and rarefaction largely in their surroundings. Hence it's necessary to know the behavior of these waves to protect destructive effects. The aim of this work to observe the propagation of shock and rarefaction waves in various dynamics due to solve non-linear hyperbolic inviscid Burgers' equation numerically. The models adopted here two numerical schemes which enable us to solve non-linear hyperbolic Burgers' equation numerically. The first order explicit upwind scheme (EUDS) and second order Lax-Wendroff schemes are used to solve this equation to improve our understanding of the numerical diffusion (smearing) and oscillations that can be present when using such schemes. In order to understand the behavior of the solution we use method of characteristics to find the exact solution of inviscid Burgers' equation. Numerical solutions are studied for different initial conditions and the shock and rarefaction waves are investigated for Riemann problem. We present stability analysis of the schemes and establish stability condition which leads to determine time step selection in terms of spatial step size with maximum initial value. Numerical result for these schemes are compared with an exact solution of inviscid Burgers' equation in terms of accuracy by error estimation. The numerical features of the rate of convergence are presented graphically. This analysis helps us to understand a wide range of physical phenomenon of the properties of wave as well as saves in several aspects in real life.

**Keywords:** Shock and Rarefaction, Burgers' Equation, Explicit Upwind and Lax-Wendroff Schemes, Rankine-Hugoniot Jump Condition, Riemann Problem

## 1. Introduction

In the area of continuum mechanics, shock waves are a mass phenomenon; based on matter their frequent occurrence are more or less compressible. In the state of the medium, large disturbances in a compressible medium propagate supersonically as abrupt alters. In real life, shock waves surround everyday humans. In nature they are generated by lightning, earthquakes, volcanic eruptions, and meteorite impact. Even in the earth, shielded by its own magnetic field. Shocks come into existence by artificial generation in several ways, such as with chemical or nuclear explosions, with the sonic boom of supersonic flying projectile and any supersonic aircraft, by a bullet pushing the air in the barrel of a rifle. These shock waves, which attached at the body is either steady waves, or unsteady ones, which change their place with passing time. All sorts of the mentioned shock waves as well as rarefaction can have destructive effects, hence steps must be undertaken to minimize them. Therefore we need to know the physical phenomenon of these waves. From the theoretical point of view shock waves as well as rarefaction are a good example of nonlinear wave propagations [1]. In applied mathematics, nonlinear partial differential equations have an important place to model and analyze these in real-world physical problems [2]. Therefore, the classical Burgers' equation which is in the class of nonlinear partial differential equations that has been a center of interest for researchers studying various physical phenomena such as theory of shock waves, fluid dynamics, turbulent flow and gas dynamics [3-5]. This equation is one of the most useful formulations of the behavior of the shock waves in which nonlinear advection and diffusion can be observed [6]. The Burgers' equation

$$u_t + uu_x = \vartheta u_{xx} \tag{1}$$

is firstly studied by Harry Bateman (in 1915) who come up with its steady state solutions and Burgers' explained it is a mathematical model for turbulent flow. Hope and Cole separately showed afterwards that it can be transformed into linear heat equation [3, 5-8]. In recent years, the Burgers' equation continued to draw the attention of researchers. It is used as a model to test several numerical methods since it includes a convection term $uu_x$ and a viscosity term $\vartheta u_{xx}$. In fact, Burgers' equation represents one dimensional Navier-Stokes equation when pressure and force terms are dropped from Navier-Stokes equation [6, 9]. Another importance of this equation is that it allows us to compare the quality of numerical method applied to a nonlinear equation.

When the limit $\vartheta \to 0$ equation (1.1) becomes a hyperbolic equation, called the inviscid Burgers' equation

$$u_t + uu_x = 0 \tag{2}$$

This limiting equation is important because it provides a simple example of a conservation law [10]. A first order partial differential equation for $u(t, x)$ is called a conservation law, if it can be written in the form

$$u_t + f(u)_x = 0 \tag{3}$$

For equation (3), $f(u) = {u^2}/{2}$ exhibit the formation of shock which appear in the solution after a finite time and then propagating in a regular manner [11]. Based on the above literature studies, we motivate to investigate further efficient finite difference schemes for the numerical solution of inviscid Burgers' equation.

In this paper, we consider the Burgers' equation in inviscid form. The inviscid Burgers' equation serves as a basic case study for more complex nonlinear wave equations since it has the properties of nonlinear conservation laws [12]. Here, we use the weak solution concept as a result of the Riemann problem where shock and rarefaction waves are observed. Next, the numerical schemes explicit upwind scheme and Lax-Wendroff numerical scheme are perform for inviscid Burgers' equation. Analyzed the stability conditions and efficiency of those schemes for time step selections. We compare the schemes with exact (weak) solution of the Riemann problem and estimate the relative error for two schemes in order to show the rate of convergence. This is the first time where we implement such numerical schemes for the simulation of shock and rarefaction waves.

The rest part of this paper is organized as: Section 2 includes the method and materials; Section 3 represents the results and discussion; and Section 4 summarizes the concluding remarks.

## 2. Methods and Materials

### *2.1. Shock*

A shock wave is surface of discontinuity propagation in a gas at which density and velocity experience abrupt changes. One can imagine two type shock waves: (positive) compression shocks which propagates into the direction where the density of the gas is a minimum, and (negative) rarefaction waves which propagates into the direction of maximum density.

At the beginning of the simulation we disturb the flow with a velocity such as:

$$u(0, x) = u_0 sin\left(\frac{2\pi x}{L}\right) \tag{4}$$

The fasted fluid catches up the slowest one so that to create a velocity break. This phenomenon is called shock.

We can notice that if the disturbance was:

$$u(0, x) = u_0 sin\left(\frac{2\pi x}{L} + \pi\right) \tag{5}$$

The contrary would have happen and the slope would have decreased [13].

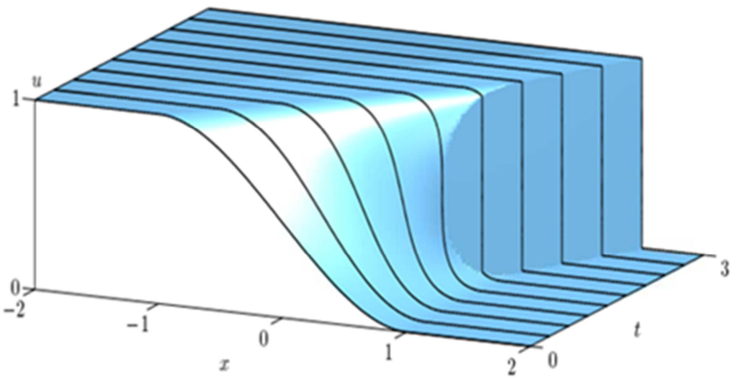


***Figure 1.*** *Formation of shock.*

### 2.2. Shock Curve

A propagating wave demarcating the path at which densities and velocities are discontinuous is called the shock curve [14]. Shock, which appears in the solution after a finite time and then propagating in a regular manner. Figure 1 shows an example.

#### 2.2.1. Exact Solution of Inviscid Burgers' Equation

To solve the inviscid Burgers' equation analytically we consider Riemann initial value problem having piecewise constant functions given in the following form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, x \in \mathbb{R}, t \in \mathbb{R}^{+} \tag{6}$$

$$u(0,x) = \begin{cases} u_L \; if \; x < 0, \\ u_R \; if \; x > 0. \end{cases}$$

The problem consists of two parts depending on the values of $u_L$ and $u_R$.

Case I ($\mathrm{u_L} > \mathrm{u_R}$): The initial condition is

$$u\,(0,x) = \begin{cases} 1 \; if \; x \le 0, \\ 0 \; if \; x > 0. \end{cases}$$

The solution of (4) is

$$u(t,x) = \begin{cases} 1 \; if \; x \le st \\ 0 \; if \; x > st. \end{cases}$$

where, 's' is the shock speed, the speed at which the discontinuity travels.

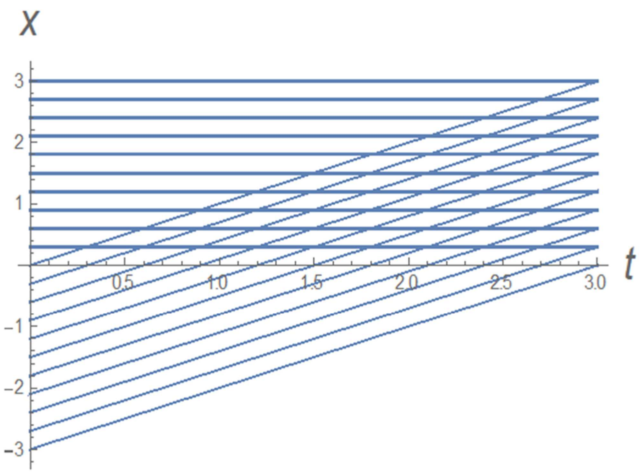


***Figure 2.*** *Characteristic curve.*

$$s = \frac{u_L + u_R}{2} = \frac{1}{2}$$

is obtained by the Rankine-Hugoniot jump condition [15, 16].

Case II ($\mathrm{u_L} < \mathrm{u_R}$): The initial condition is

$$u_0(x) = \begin{cases} 0 \; if \; x \le 0, \\ 1 \; if \; x > 0. \end{cases}$$

The solution of (4) is

$$u(t,x) = \begin{cases} 0 \; if \; x \le st \\ 1 \; if \; x > st. \end{cases}$$

where the propagation speed is

$$s = \frac{u_L + u_R}{2} = \frac{1}{2}$$

is obtained by the Rankine-Hugoniot jump condition.

The characteristic curve is

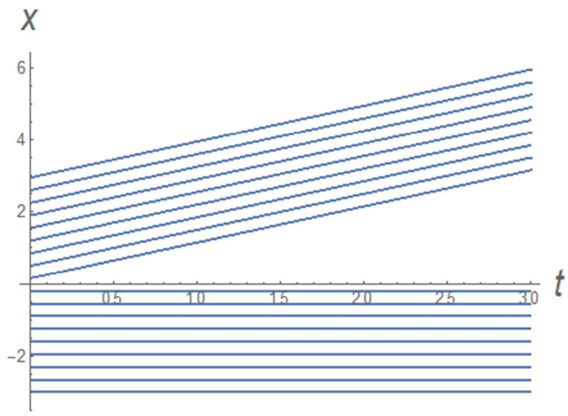


***Figure 3.*** *Characteristic curve.*

#### 2.2.2. Numerical Solution of Inviscid Burgers' Equation

In order to implement the numerical finite difference method, we discretize the plane with mesh grid size $\Delta x \times \Delta t$. The grid width and time steps are taken individually. The temporal and spatial coordinate at grid point $u(x_j, t^n)$ is defined as

$$x_j = x_0 + j\Delta x, j = 0,1, \dots \dots \dots, M$$

$$t^n = t^0 + n\Delta t, n = 0,1, \dots \dots \dots, N$$

The approximate solution at a discrete set of points

$$u(x_j, t^n) = u_j^n$$

Using Taylor's series expansion, we discretize the time derivative by forward difference formula

$$\left.\frac{\partial u}{\partial t}\right|_{(t^n, x_j)} \approx \frac{u_j^{n+1} - u_j^n}{\Delta t} \tag{7}$$

The spatial derivative by the first order backward difference formula

$$\left.\frac{\partial u}{\partial x}\right|_{(t^n, x_j)} \approx \frac{u_j^n - u_{j-1}^n}{\Delta t} \tag{8}$$

The spatial derivative by the 2nd order central difference formula

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} \tag{9}$$

Substituting equation (5) and (6) into (4) we have explicit upwind scheme in conservative form is

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(\left(u_j^n\right)^2 - \left(u_{j-1}^n\right)^2\right)$$

And non-conservative form is

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(u_j^n\left(u_j^n - u_{j-1}^n\right)\right) \tag{10}$$

We can drive the Lax-wendroff scheme using the modified equation of the non-conservative form:

Since $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

$$\Rightarrow \frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial t}\left(u\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = -\frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial x}\right) - u\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = -\left(-u\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x} - u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = u\left(\frac{\partial u}{\partial x}\right)^2 - u\frac{\partial}{\partial x}\left(-u\frac{\partial u}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = u\left(\frac{\partial u}{\partial x}\right)^2 + u\left(\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = u\left(\frac{\partial u}{\partial x}\right)^2 + u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = 2u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2}$$

Now $u\left(x_j, t^{n+1}\right) = u\left(x_j, t^n\right) + \Delta t\frac{\partial u\left(x_j, t^n\right)}{\partial t}$

$$+\frac{(\Delta t)^2}{2}\frac{\partial^2 u\left(x_j, t^n\right)}{\partial t^2} + (\Delta t)^3$$

$$\Rightarrow u\left(x_j, t^{n+1}\right)$$

$$= u\left(x_j, t^n\right) - \Delta t.u\left(x_j, t^n\right)\frac{\partial u\left(x_j, t^n\right)}{\partial x}$$

$$+\frac{(\Delta t)^2}{2}\left[2u\left(x_j, t^n\right)\left(\frac{\partial u\left(x_j, t^n\right)}{\partial x}\right)^2 + \left(u\left(x_j, t^n\right)\right)^2\frac{\partial^2 u\left(x_j, t^n\right)}{\partial x^2}\right]$$

$$\Rightarrow u_j^{n+1} = u_j^n - \frac{\Delta t}{2\Delta x}\left(u_j^n\right)\left(u_{j+1}^n - u_{j-1}^n\right)$$

$$+\frac{(\Delta t)^2}{2}\left[2\left(u_j^n\right)\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right)^2 + \left(u_j^n\right)^2\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}\right)\right] \tag{11}$$

Which is the required Lax-Wendroff scheme for inviscid Burgers' equation in non-conservative form.

# 3. Numerical Results and Discussion

### 3.1. Stability Analysis

By the convex combination we obtain the stability condition of EUDS and Lax-Wendroff Scheme. Equation (10) and (11) implies that the new solution is a convex combination of the previous solutions. That is the new solution at new time step $n+1$ is an average of the solutions at the previous time step at the spatial nodes $i+1, i$ *and* $i-1$.

The stability conditions of inviscid Burgers' equation for EUDS and Lax-Wenfroff scheme are as follows:

***Table 1.*** *Stability conditions for two Schemes.*

| Schemes | Order of Accuracy | Stability conditions |
| --- | --- | --- |
| EUDS | $O(\Delta t, \Delta x)$ | $0 \le max_{j,n}\lvert u_j^n\rvert\frac{\Delta t}{\Delta x} \le 1$ |
| Lax-Wendroff | $O(\Delta t^2, \Delta x^2)$ | $-1 \le \max_j\{u_j^n\}\frac{\Delta t}{\Delta x} \le 1$ |

### *3.2. Relative Error for Explicit Upwind and Lax-Wendroff Scheme*

We present finite difference schemes for $u(t, x)$ up to time $t = 20$ second in temporal grid size $\Delta t = 0.0025$ in spatial domain [0,30] with spatial grid size $\Delta x = 0.10$ which satisfy the stability condition.

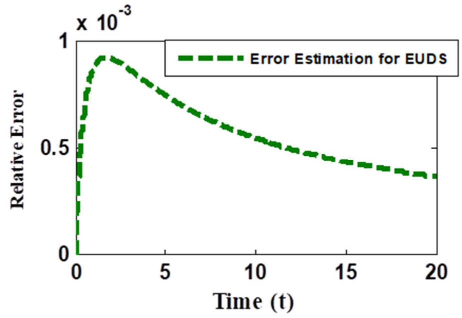


***Figure 4.*** *Relative Error of inviscid Burgers' Equation for EUDS.*

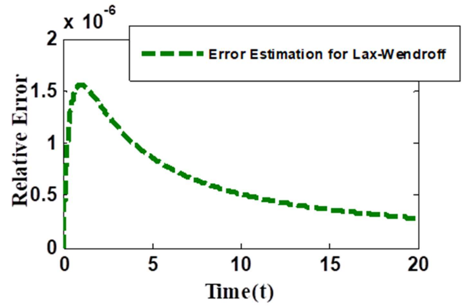


***Figure 5.*** *Relative Error of inviscid Burgers' Equation for Lax-Wendroff Scheme.*

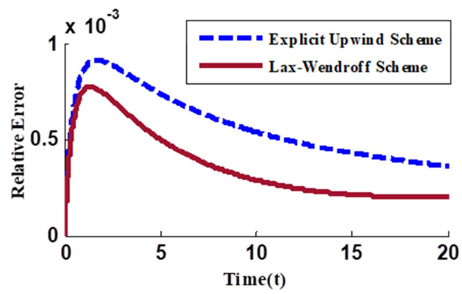


***Figure 6.*** *Comparison of relative error for EUDS and Lax-Wendroff scheme.*

Figures 4 and 5, we present Relative Error by using EUDS and Lax-Wendroff scheme for $u(t, x)$ up to time $t = 20$ second in temporal grid size $\Delta t = 0.0025$ in spatial domain [0,30] with spatial grid size $\Delta x = 0.10$. Figure 6 shows the comparison of relative error for the two finite difference schemes. The relative error for EUDS remain below 0.0363 and Lax-Wendroff remain below 0.000028325. From these figure, we found that Lax-Wendroff scheme provides more accurate result than EUDS.

### *3.3. Convergence of Relative Error*

The convergence of relative error by the scheme EUDS and Lax-Wendroff scheme are shown in here. The error for different temporal sizes are computed as presented in the following figures 6 to 8.

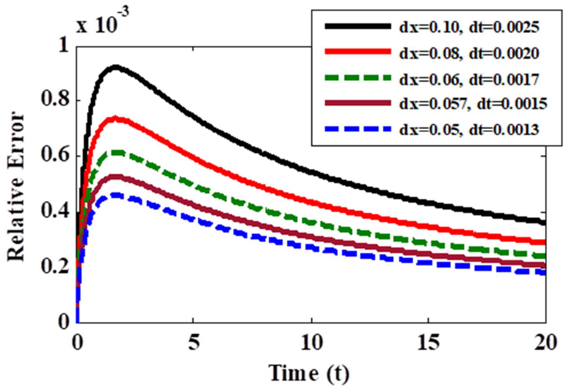


***Figure 7.*** *Convergence of Relative Error for EUDS.*

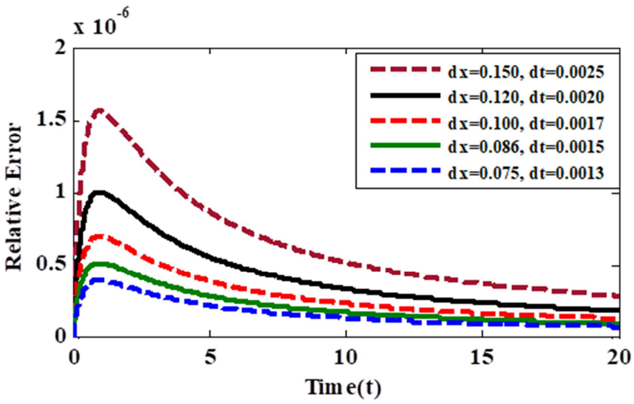


***Figure 8.*** *Convergence of Relative Error for Lax-Wendroff Scheme.*

We observe that error reduces for smaller $\Delta t$ and $\Delta x$ EUDS and Lax-Wendroff schemes are shows good rate of convergence. We therefore apply these scheme to show nonlinear phenomena shock and rarefaction which occur in our real life.

### *3.4. Numerical Results for Shock and Rarefaction*

In this section, numerical shock and rarefaction are holding for the initial value problem of the inviscid Burgers' equation. We compare our model to the classical and numerical solutions. Hence, the first and second order finite difference approximation methods are used. For the EUDS and Lax-Wendroff schemes, we examine Riemann initial value problems with the shock and rarefaction waves.

Figures 9 to 16 were drawn using EUDS and Lax-Wendroff method with a step size $\Delta t = 0.0250$ and $\Delta x = 0.0333$ with a number of iterations in time as $t = 0\,sec$, $t = 5\,sec$, $t = 10\,sec$, $t = 15\,sec$ and $t = 20\,sec$ respectively. We can observe that for $t = 0\,sec$ there is no oscillation but at $t = 5\,sec$ oscillations shows at the corner points where the solution is not smooth, unlike the explicit upwind method. Observe that the scheme remain stable in spite of oscillation. However, the illustration of shock and rarefaction waves with a number of iterations in time as $t = 0\,sec$, $t = 5\,sec$, $t = 10\,sec$, $t = 15\,sec$ and $t = 20\,sec$ is quite similar to the one in explicit upwind method.

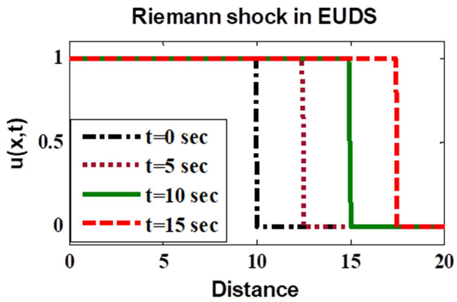


***Figure 9.*** *Explicit Upwind method for shock solution.*

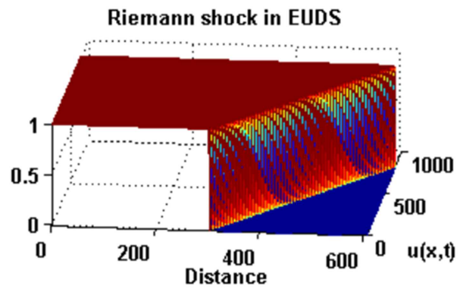


***Figure 10.*** *Explicit Upwind method for shock solution in mesh form.*

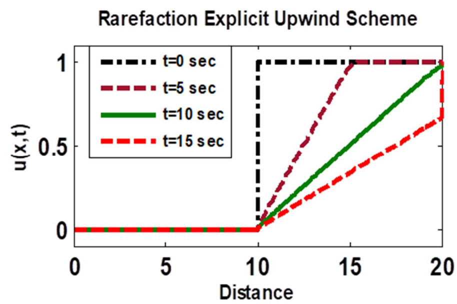


***Figure 11.*** *Explicit Upwind method for rarefaction solution.*

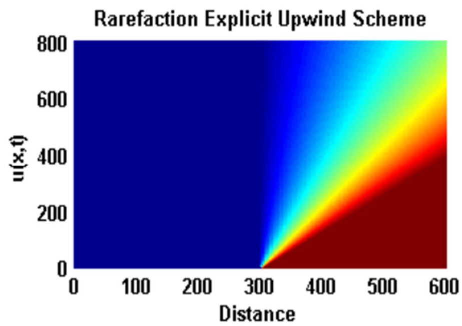


***Figure 12.*** *Explicit Upwind method for rarefaction solution.*

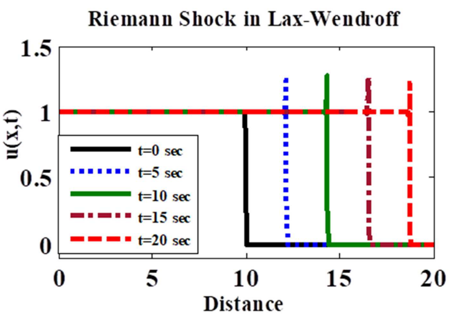


*Figure 13.* *Lax-Wendroff method for shock solution.*

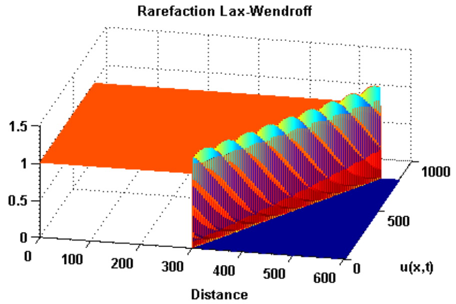


*Figure 14.* *Lax-Wendroff method for shock solution in mesh form.*

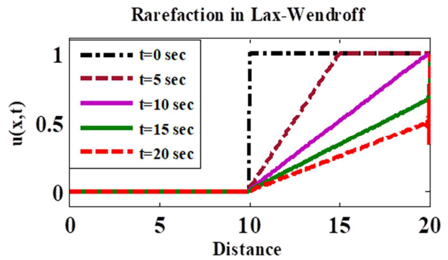


*Figure 15.* *Lax-Wendroff method for rarefaction solution.*

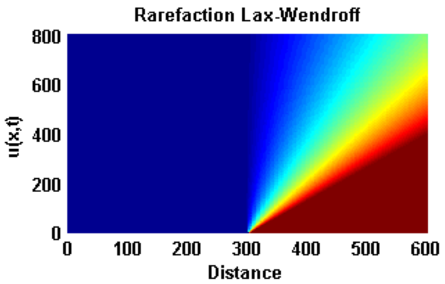


*Figure 16.* *Lax-Wendroff method for rarefaction solution in mesh form.*

### *3.5. Comparison Between Exact Solution of Riemann Problem with $u_L = 1$ and $u_R = 0$ Numerical Schemes*

The solution for this data describes a shock which is propagating in the positive x-direction, with a speed of 0.5 $m/s$. We can compare the behavior of the numerical schemes, since the analytic solution is known. Figures 17 and 18 were plotted using a step size $\Delta x = 0.5000$ and a time step $\Delta t = 0.0125$, with a number of 80 time steps. At this particular time point the shock has moved to $x = 40$ from its initial position. By analyzing Figures 17 and 18 we can clearly see that the first order explicit upwind scheme has introduced numerical diffusion (smearing). The smearing for the first order upwind scheme, this being a direct feature of the truncation error terms of the scheme, whilst the second order Lax-Wendroff scheme are much more accurate at capturing the shock. The Lax-Wendroff scheme produces spurious oscillations to the left of the shock (i.e. corner point) as can be seen by looking at Figures 17 and 18. These oscillations can be gradually reduce by taking step size $\Delta x$ as small.

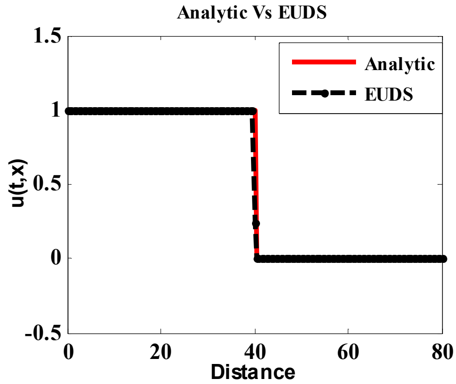


*Figure 17.* *Comparison of Riemann weak Vs explicit upwind scheme.*

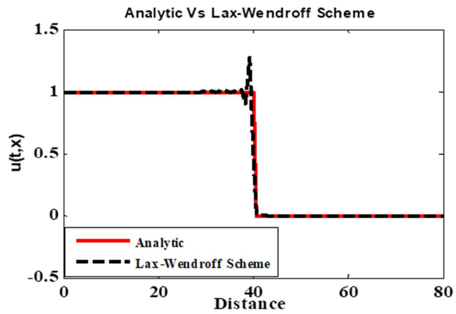


*Figure 18.* *Comparison of Riemann weak Vs Lax-Wendroff scheme.*

## 4. Conclusion

This research studied the inviscid Burgers' equation analytically and numerically. Analytic form of the solution for this equation is explained by the method of characteristics. We studied weak solution of inviscid Burgers' equation as Riemann problem. We have shown the derivation of the explicit finite difference schemes and analyzed the stability of these schemes. We computed the relative errors for the different schemes which shown a very good rate of convergence. We have shown the comparison between explicit upwind and Lax-Wendroff schemes with exact solution of Riemann problem as well. We observed that the first order scheme is the most dissipative method compared to second order scheme. The Lax-Wendroff scheme exhibited oscillatory motion close the corner point where shock forms. The two schemes illustrated the behavior of shock and rarefaction waves depending on the sort of the initial conditions. In this paper, we have looked on that the inviscid Burgers' equation is a significant model to denominate the shock and rarefaction waves.

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